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MATH PHENOMENON

ÎNVĂȚARE DE EXCELENȚĂ
supersucces

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Chapter 1 Famous Inequalities

Cauchy–Schwarz Inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 < \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Minkowski Inequality

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}; \text{ for } p \ge 1.$$

Hölder's Inequality

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \text{ for } p, q, > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Bernoulli Inequality

$(1 + x)^r \ge 1 + rx$ for $x \ge 1$, $r \in \mathbb{R} \setminus (0, 1)$. Reverse for $r \in [0, 1]$.

$(1 + x)^r \le 1 + (2^r - 1)x$ for $x \in [0, 1]$, $r \in \mathbb{R} \setminus (0, 1)$.

$$(1 + x)^n \le \frac{1}{1 - nx} \text{ for } x \in [-1, 0],\ n \in \mathbb{N}.$$

$$(1 + x)^r \le 1 + \frac{rx}{1 - (r - 1)x} \text{ for } x \in \left[-1, \frac{1}{r - 1}\right),\ r > 1.$$

$(1 + nx)^{n+1} \ge (1 + (n + 1)x)^n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$.

$(a + b)^n \le a^n nb(a + b)^{n-1}$ for $a, b \ge 0$, $n \in \mathbb{N}$.

$$\left(1 + \frac{x}{p}\right)^p \ge \left(1 + \frac{x}{q}\right)^q \text{ for } (i)\ x > 0, p > q > 0,$$

(ii) $-p < -q < x < 0$, (iii) $-q > -p > x > 0$. Reverse for:
(iv) $q < 0 < p$, $-p > x > 0$, (v) $q < 0 < p$, $-p < x < 0$.

Exponential Inequalities

$$e^x \ge \left(1 + \frac{x}{n}\right)^n \ge 1 + x,\ \left(1 + \frac{x}{n}\right)^n \ge e^x\left(1 - \frac{x^2}{n}\right) \text{ for } n > 1,\ |x| \le n.$$

$$e^x \ge x^e \text{ for } x \in \mathbb{R}, \text{ and } \frac{x^n}{n!} + 1 \le e^x \le \left(1 + \frac{x}{n}\right)^{n + \frac{x}{2}} \text{ for } x, n > 0.$$

$e^x \geq 1 + x + \frac{x^2}{2}$ for $x \geq 0$, reverse for $x \leq 0$.

$e^{-x} \leq 1 - \frac{x}{2}$ for $x \in [0, \sim 1.59]$ and $2^{-x} \leq 1 - \frac{x}{2}$ for $x \in [0, 1]$.

$\frac{1}{2-x} < x^x < x^2 - x + 1$ for $x \in (0, 1)$.

$x^{\frac{1}{r}}(x-1) \leq rx(x^{\frac{1}{r}} - 1)$ for $x, r \geq 1$.

$x^y + y^x > 1$ and $e^x > \left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$ for $x, y > 0$.

$2 - y - x^{-x-y} \leq 1 + x \leq y + e^{x-y}$, and $e^x \leq x + e^{x^2}$ for $x, y \in \mathbb{R}$.

Logarithm Inequalities

$\frac{x-1}{x} \leq \ln(x) \leq \frac{x^2 - 1}{2x} \leq x - 1$, $\ln(x) \leq n\left(x^{\frac{1}{n}} - 1\right)$ for $x, n > 0$.

$\frac{2x}{2+x} \leq \ln(1 + x) \leq \frac{x}{\sqrt{x+1}}$ for $x \geq 0$, reverse for $x \in (-1, 0]$.

$\ln(n + 1) < \ln(n) + \frac{1}{n} \leq \sum_{i=1}^{n} \frac{1}{i} \leq \ln(n) + 1$.

$\ln(1 + x) \geq \frac{x}{2}$ for $x \in [0, \sim 2.51]$, reverse elsewhere.

$\ln(1 + x) \geq x - \frac{x^2}{2} + \frac{x^3}{4}$ for $x \in [0, \sim 0.45]$, reverse elsewhere.

$\ln(1 - x) \geq -x - \frac{x^2}{2} + \frac{x^3}{2}$ for $x \in [0, \sim 0.43]$, reverse elsewhere.

Trigonometric Inequalities

$$x - \frac{x^3}{2} \leq x \cos x \leq \frac{x \cos x}{1 - \frac{x^2}{3}} \leq x\sqrt[3]{\cos x} \leq x - \frac{x^3}{6} \leq x \cos \frac{x}{\sqrt{3}} \leq \sin x.$$

Hyperbolic Inequalities

$$x \cos x \leq \frac{x^3}{\sinh^2 x} \leq x \cos^2\left(\frac{x}{2}\right) \leq \sin x \leq \frac{(x \cos x + 2x)}{3} \leq \frac{x^2}{\sinh x'}.$$

$$\frac{2}{\pi}x \leq \sin x \leq x \cos\left(\frac{x}{2}\right) \leq x \leq x + \frac{x^3}{3} \leq \tan x \text{ all for } x \in \left[0, \frac{\pi}{2}\right].$$

$\cosh(x) + \alpha \sinh(x) \leq e^{x\left(\alpha + \frac{x}{2}\right)}$ for $x \in \mathbb{R}$, $\alpha \in [-1, 1]$.

Binomial Inequalities

$$\max\left\{\frac{n^k}{k^k}, \frac{(n-k+1)^k}{k!}\right\} \le \binom{n}{k} \le \frac{n^k}{k!} \le \frac{(en)^k}{k^k} \text{ and } \binom{n}{n} \le \frac{n^n}{k^k(n-k)^{n-k}} \le 2^n.$$

$$\frac{n^k}{4k!} \le \binom{n}{k} \text{ for } \sqrt{n} \ge k \ge 0 \text{ and } \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right) \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{9n}\right).$$

$$\binom{n_1}{k_1}\binom{n_2}{k_2} \le \binom{n_1+n_2}{k_1+k_2} \text{ for } n_1 \ge k_1 \ge 0,\ n_2 \ge k_2 \ge 0.$$

$$\frac{\sqrt{\pi}}{2}G \le \binom{n}{\alpha n} \le G \text{ for } G = \frac{2^{nH(\alpha)}}{\sqrt{2\pi n\alpha(1-\alpha)}},\ H(x) = -\log_2(x^x(1-x)^{1-x}).$$

$$\sum_{i=0}^{d}\binom{n}{i} \le n^d + 1 \text{ and } \sum_{i=0}^{d}\binom{n}{i} \le 2^n \text{ for } n \ge d \ge 0.$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \left(\frac{en}{d}\right)^d \text{ for } n \ge d \ge 1.$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \binom{n}{d}\left(1+\frac{d}{n-2d+1}\right) \text{ for } \frac{n}{2} \ge d \ge 0.$$

$$\binom{n}{\alpha n} \le \sum_{i=0}^{\alpha n}\binom{n}{i} \le \frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n} \text{ for } \alpha \in \left(0, \frac{1}{2}\right).$$

Square Root Inequality

$$2\sqrt{x+1} - 2\sqrt{x} < \frac{1}{\sqrt{x}} < \sqrt{x+1} - \sqrt{x-1} < 2\sqrt{x} - 2\sqrt{x-1} \text{ for } x \ge 1.$$

Stirling Inequality

$$e\left(\frac{n}{e}\right)^n \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{(12n+1)}} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \le en\left(\frac{n}{e}\right)^n.$$

Means Inequality

$$\min\{x_i\} \le \frac{n}{\sum_i x_i^{-1}} \le \left(\prod_i x_i\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_i x_i \le \sqrt{\frac{1}{n}\sum_i x_i^2} \le \max\{x_i\}.$$

Power Means Inequality

$M_p \le M_q$ for $p \le q$, where $M_p = \left(\sum_i w_i |x_i|^p\right)^{\frac{1}{p}}$, $w_i \ge 0$, $\sum_i w_i = 1$.

In the limit $M_0 = \prod_i |x_i|^{w_i}$, $M_{-\infty} = \min_i\{x_i\}$, $M_\infty = \max_i\{x_i\}$.

Lehmer Inequality

$$\frac{\sum_i w_i |x_i|^p}{\sum_i w_i |x_i|^{p-1}} \le \frac{\sum_i w_i |x_i|^q}{\sum_i w_i |x_i|^{q-1}} \text{ for } p \le q,\ w_i \ge 0.$$

Log Mean Inequality

$$\sqrt{xy} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)(xy)^{\frac{1}{4}} \le \frac{x-y}{\ln(x)-\ln(y)} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \le \frac{x+y}{2} \text{ for } x, y > 0.$$

Heinz Inequality

$$\sqrt{xy} \le \frac{x^{1-\alpha}y^{\alpha}+x^{\alpha}y^{1-\alpha}}{2} \le \frac{x+y}{2} \text{ for } x, y > 0,\ \alpha \in [0, 1].$$

Maclaurin–Newton Inequality

$$S_k^2 \ge S_{k-1}S_{k+1} \text{ and } \sqrt[k]{S_k} \ge \sqrt[(k+1)]{S_{k+1}} \text{ for } 1 \le k \le n,\ S_k = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1} a_{i_2} \ldots a_{i_k},\ a_i \ge 0.$$

Jensen's Inequality

$\varphi\left(\sum_i p_i x_i\right) \le \sum_i p_i \varphi(x_i)$ where $p_i > 0$, $\sum p_i = 1$, and φ convex.

Alternatively: $\varphi(E[X]) \le E[\varphi(X)]$. For concave φ the reverse holds.

Chebyshev Inequality

$$\sum_{i=1}^{n} f(a_i)g(b_i)p_i \ge \left(\sum_{i=1}^{n} f(a_i)p_i\right)\left(\sum_{i=1}^{n} g(b_i)p_i\right) \ge \sum_{i=1}^{n} f(a_i)g(b_{n-1+i})p_i \text{ for } a_1 \le \ldots \le a_n,$$

$b_1 \le \ldots b_n$ and f, g nondecreasing, $p_i \ge 0$, $\sum p_i = 1$.

Alternatively: $E[f(X)g(X)] \ge E[f(X)]E[g(X)]$.

Rearrangement Inequality

$\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i b_{\pi(i)} \ge \sum_{i=1}^{n} a_i b_{n-i+1}$ for $a_1 \le \ldots \le a_n$, $b_1 \le \ldots \le b_n$ and π a permutation of $[n]$.

More generally:

$\sum_{i=1}^{n} f_i(b_i) \ge \sum_{i=1}^{n} f_i(b_{\pi(i)}) \ge \sum_{i=1}^{n} f_i(b_{n-i+1})$ with $(f_{i+1}(x) - f_i(x))$ nondecreasing for all $1 < i < n$.

Weierstrass Inequality

$\prod_i (1-x_i)^{w_i} \ge 1 - \sum_i w_i x_i$ where $x_i \le 1$, and either $w_i \ge 1$ (for all i) or $w_i \le 0$ (for all i).

If $w_i \in [0, 1]$, $\sum w_i \le 1$ and $x_i \le 1$, the reverse holds.

Young Inequality

$$\left(\frac{1}{px^p}+\frac{1}{qx^q}\right)^{-1} \le xy \le \frac{x^p}{p}+\frac{y^q}{q} \text{ for } x, y \ge 0, p, q > 0,\ \frac{1}{p}+\frac{1}{q}=1.$$

Kantorovich Inequality

$(\Sigma_i x_i^2)(\Sigma_i y_i^2) \le \left(\frac{A}{G}\right)^2 (\Sigma_i x_i y_i)^2$ for $x_i, y_i > 0$, $0 < m \le \frac{x_i}{y_i} \le M < \infty$, $A = \frac{(m+M)}{2}$, $G = \sqrt{mM}$.

Sum–integral Inequality

$\int_{L-1}^{U} f(x)dx \le \sum_{i=L}^{U} f(i) \le \int_{L}^{U+1} f(x)dx$ for f nondecreasing.

Cauchy Inequality

$\varphi'(a) \le \frac{f(b)-f(a)}{b-a} \le \varphi'(b)$ where $a < b$, and φ convex.

Hermite-Hadamard Inequality

$\varphi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \varphi(x)dx \le \frac{\varphi(a)+\varphi(b)}{2}$ for φ convex.

Chong Inequality

$\sum_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \ge n$ and $\prod_{i=1}^{n} a_i^{a_i} \ge \prod_{i=1}^{n} a_i^{a_{\pi(i)}}$ for $a_i > 0$.

Gibbs' Inequality

$\Sigma_i a_i \log\frac{a_i}{b_i} \ge a\log\frac{a}{b}$ for $a_i, b_i \ge 0$, or more generally:

$\Sigma_i a_i\varphi\left(\frac{b_i}{a_i}\right) \le a\varphi\left(\frac{b}{a}\right)$ for φ concave, and $a = \Sigma a_i$ $b = \Sigma b_i$.

Shapiro's Inequality

$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \ge \frac{n}{2}$ where $x_i > 0$, $(x_{n+1}, x_{n+2}) = (x_1, x_2)$.

Hadamard's Inequality

$(\det A)^2 \le \prod_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2$ where A is an $n \times n$ matrix.

Schur's Inequality

$\sum_{i=1}^{n} \lambda_i^2 \le \sum_{i,j=1}^{n} A_{ij}^2$ and $\sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} \lambda_i$ for $1 \le k \le n$.

A is an $n \times n$ matrix. For the second inequality A is symmetrical. $\lambda_1 \ge \ldots \ge \lambda_n$ the eigen values, $d_1 \ge \ldots \ge d_n$ the diagonal elements.

Ky Fan Inequality

$\frac{\prod_{i=1}^{n} x_i^{a_i}}{\prod_{i=1}^{n}(1-x_i)^{a_i}} \le \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i(1-x_i)}$ for $x_i \in \left[0,\frac{1}{2}\right]$, $a_i \in [0, 1]$, $\sum a_i = 1$.

Aczél Inequality

$\left(a_1b_1-\sum_{i=2}^{n}a_ib_i\right)^2\ge\left(a_1^2-\sum_{i=2}^{n}a_i^2\right)\left(b_1^2-\sum_{i=2}^{n}b_i^2\right)$ given that $a_1^2>\sum_{i=2}^{n}a_i^2$ or $b_1^2>\sum_{i=2}^{n}b_i^2$.

Mahler Inequality

$\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}\ge\prod_{i=1}^{n}x_i^{\frac{1}{n}}+\prod_{i=1}^{n}y_i^{\frac{1}{n}}$ where $x_i, y_i > 0$.

Abel Inequality

$b_1\min_k\sum_{i=1}^{k}a_i\le\sum_{i=1}^{n}a_ib_i\le b_1\max_k\sum_{i=1}^{k}a_i$ for $b_1\ge\ldots\ge b_n\ge0$.

Milne Inequality

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\le\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right).$$

Hardy–Carleman Inequality

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k}|a_i|\right)^{\frac{1}{k}}\le e\sum_{k=1}^{n}|a_k|.$$

Sum & Product Inequality

$\sum_{j=1}^{m}\prod_{i=1}^{n}a_{ij}\ge\sum_{j=1}^{m}\prod_{i=1}^{n}a_{i\pi(j)}$ and $\sum_{j=1}^{m}\prod_{i=1}^{n}a_{ij}\le\sum_{j=1}^{m}\prod_{i=1}^{n}a_{i\pi(j)}$ where $0\le a_{i1}\le\ldots a_{im}$ for $i=1,\ldots,n$ and π is a permutation of $[n]$.

$\left|\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i\right|\le\sum_{i=1}^{n}|a_i-b_i|$ for $|a_i|, |b_i|\le1$.

$\prod_{i=1}^{n}(\alpha+a_i)\ge(1+\alpha)^n$, where $\prod_{i=1}^{n}a_i\ge1$, $a_i>0$, $\alpha>0$.

Callebaut Inequality

$\left(\sum_i a_i^{1+x}b_i^{1-x}\right)\left(\sum_i a_i^{1-x}b_i^{1+x}\right)\ge\left(\sum_i a_i^{1+y}b_i^{1-y}\right)\left(\sum_i a_i^{1-y}b_i^{1+y}\right)$ for $1\ge x\ge y\ge0$, and $i=1,\ldots,n$.

Karamata's Inequality

$\sum_{i=1}^{n}\varphi(a_i)\ge\sum_{i=1}^{n}\varphi(b_i)$ for $a_1\ge a_2\ge\ldots\ge a_n$ and $b_1\ge\ldots\ge b_n$, and $\{a_i\}\ge\{b_i\}$ (majorization), i.e. $\sum_{i=1}^{t}a_i\ge\sum_{i=1}^{t}b_i$ for all $1\le t\le n$, with equality for $t=n$ and φ is convex (for concave φ the reverse holds).

Muirhead Inequality

$\frac{1}{n!}\sum_\pi x_{\pi(1)}^{a_1}\ldots x_{\pi(n)}^{a_n}\ge\frac{1}{n!}\sum_\pi x_{\pi(1)}^{b_1}\ldots x_{\pi(n)}^{b_1}$ where $a_1\ge a_2\ge\ldots\ge a_n$ and $b_1\ge b_2\ge\ldots\ge b_n$ and $\{a_k\}\ge\{b_k\}$, $x_i\ge0$ and the sums extend over all permutations π of $[n]$.

Huygens Inequality

$\prod_{k=1}^{n}(1+x_k)\ge(1+\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n})^n$, $n\in\mathbb{N}^*\setminus\{1\}$; $x_k\in\mathbb{R}$; $k=\overline{1,n}$.

Bernoulli Inequality Generalized

$\prod_{k=1}^{n}(1+x_k)\geq 1+\sum_{k=1}^{n}x_k$, $n\in\mathbb{N}^*$; $x_k\in[-1,\infty)$; $k=\overline{1,n}$; $x_ix_j\in\mathbb{R}_+$, for any $i,j\in\overline{1,n}$.

Tiberiu Popoviciu Inequality

$$\frac{1}{3}(f(x)+f(y)+f(z))+f\left(\frac{x+y+z}{3}\right)\geq\frac{2}{3}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right)$$

$f:[a,b]\to\mathbb{R}$ convex function.

Chebyshev Inequality

i) $(a_k)_{k=\overline{1,n}}$; $(b_k)_{k=\overline{1,n}}$ monotonic sequences with the same monotony:

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}b_k\right)\leq\frac{1}{n}\sum_{k=1}^{n}a_kb_k.$$

ii) $(a_k)_{k=\overline{1,n}}$; $(b_k)_{k=\overline{1,n}}$ monotonic sequences with different monotony:

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}b_k\right)\geq\frac{1}{n}\sum_{k=1}^{n}a_kb_k.$$

Diaz–Metcalf Inequality

$\sum_{k=1}^{n}b_k^2+mM\sum_{k=1}^{n}a_k^2\leq(m+M)\sum_{k=1}^{n}a_kb_k$; $a_k, b_k\in\mathbb{R}^*$; $m\leq\frac{a_k}{b_k}\leq M$; $k\in\overline{1,n}$; $n\in\mathbb{N}^*$.

Greub–Rheinboldt Inequality

$[a,A]$; $[b,B]\subset\mathbb{R}_+^*$; $x_k\in[a,A]$; $I_k\in[b,B]$, $t_k\in\mathbb{R}$; $k\in\overline{1,n}$; $n\in\mathbb{N}^*$.

$$\left(\sum_{k=1}^{n}t_kx_k^2\right)\left(\sum_{k=1}^{n}t_ky_k^2\right)\leq\frac{(ab+AB)^2}{4abAB}\left(\sum_{k=1}^{n}t_kx_ky_k\right)^2$$

Kantorovich Inequality

$\left(\sum_{k=1}^{n}t_kx_k\right)\left(\sum_{k=1}^{n}\frac{t_k}{x_k}\right)\leq\frac{(m+M)^2}{4mM}\left(\sum_{k=1}^{n}t_k\right)^2$; $[m,M]\subset(0,\infty)$; $x_k\in[m,M]$; $t_k\in(0,\infty)$;

$k\in\overline{1,n}$; $n\in\mathbb{N}^*$.

Polya–Szegö Inequality

$[a,A]$; $[b,B]\subset(0,\infty)$; for any $x_k\in[a,A]$; $y_k\in[b,B]$; $t_k\in\mathbb{R}$; $k\in\overline{1,n}$; $n\in\mathbb{N}$;

$$\left(\sum_{k=1}^{n}x_k^2\right)\left(\sum_{k=1}^{n}y_k^2\right)\leq\frac{(ab+AB)^2}{4abAB}\cdot\left(\sum_{k=1}^{n}x_ky_k\right)^2.$$

Schweitzer Inequality

$[m,M]\subset(0,\infty)$; $x_k\in[m,M]$; $k\in\overline{1,n}$; $n\in\mathbb{N}$;

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\leq\frac{(m+M)^2}{4mM}n^2$$

Bătinețu Inequality

$x_k, y_k \in \mathbb{R}^*$; $m \le \frac{y_k}{x_k} \le M$; $t_k \in (0, \infty)$; $k \in \overline{1,n}$; $n \in \mathbb{N}$;

$$\sum_{k=1}^{n} t_k x_k^2 + mM\sum_{k=1}^{n} t_k y_k^2 \le (m+M)\sum_{k=1}^{n} t_k x_k y_k$$

Hölder's Inequality Generalized

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}\right)^{w_j} \ge \sum_{i=1}^{n}\left(\prod_{j=1}^{n} x_{ij}^{w_j}\right);\ w_1 + w_2 + \ldots + w_n = 1$$

Weighted Means Inequality

$w = w_1 + w_2 + \ldots + w_n$;

$$\frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w} \ge \sqrt[w]{x_1^{w_1} x_2^{w_2} \cdot \ldots \cdot x_n^{w_n}} \ge \frac{w}{\frac{w_1}{x_1} + \frac{w_2}{x_2} + \ldots + \frac{w_n}{x_n}}$$

Convention $0^0 = 1$.

Maclaurin's Inequality

$$\frac{\sum y_i}{n} \ge \sqrt{\frac{\sum y_i y_j}{C_n^2}} \ge \sqrt[3]{\frac{\sum y_i y_j y_k}{C_n^3}} \ge \ldots \ge \sqrt[n-1]{\frac{\sum y_{i_1} y_{i_2} \cdot \ldots \cdot y_{i_{n-1}}}{C_n^{n-1}}} \ge \sqrt[n]{y_{i_1} y_{i_2} \cdot \ldots \cdot y_{i_n}}$$

$$\frac{a+b}{2} \ge \sqrt{ab};\quad \frac{a+b+c}{3} \ge \sqrt{\frac{ab+ac+ca}{3}} \ge \sqrt[3]{abc}$$

$$\frac{a+b+c+d}{4} \ge \sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}} \ge \sqrt[3]{\frac{abc+abd+bcd+acd}{4}} \ge \sqrt[4]{abcd}$$

Kurlyandchik's Inequality

$$\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} < 2\sum_{k=1}^{n} a_k;\ a_k > 0.$$

Murray Klamkin Inequality

$$\frac{\prod_{k=1}^{n}(1+a_k)}{(1+n)^n} \ge \frac{\prod_{k=1}^{n}(1-a_k)}{(n-1)^n};\ n \ge 2;\ a_k \in (0, \infty);\ k \in \overline{1,n};\ \sum_{k=1}^{n} a_k = 1.$$

Karamata's Inequality Generalized

$a_1 \ge b_1$; $a_1 + a_2 \ge b_1 + b_2$; $a_1 + a_2 + \ldots + a_{n-1} \ge b_1 + b_2 + \ldots + b_{n-1}$

$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n$; f convex

$f(a_1) + f(a_2) + \ldots + f(a_n) \ge f(b_1) + f(b_2) + \ldots + f(b_n)$

Bătinețu Inequality 2

$$\left(\sum_{k=1}^{n} t_k x_k\right)\left(\sum_{k=1}^{n} \frac{t_k}{x_k}\right)^{ab} \le \left(\frac{a+b}{1+ab}\right)^{1+ab} \cdot \left(\sum_{k=1}^{n} t_k\right)^{1+ab};$$

$a, b \in (0, \infty)$; $a < b$; $ab \in \mathbb{N}^*$; $x_k \in [a, b]$; $t_k \in \mathbb{R}_+$

Bătinețu Inequality 3

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)^{a(1+b)} \le \left(\frac{1}{1+ab+a} \cdot n\right)^{1+a+ab};$$

$a, b, n \in \mathbb{N}^*$; $a < b$; $x_k \in [a, b]$; $k \in \overline{1,n}$.

Schur's Inequalities

$$a^3 + b^3 + c^3 + 3abc \ge a^2b + b^2a + b^2c + c^2b + a^2c + c^2a$$

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \ge 0; r > 0$$

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca)$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \ge 0$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \ge 2(ab+bc+ca)$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \ge 2$$

$$a^4 + b^4 + c^4 + abc(a+b+c) \ge ab(a^2+b^2) + bc(b^2+c^2) + cd(c^2+a^2)$$

Useful Inequalities

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}};\ n \ge 2;\ \frac{\tan x}{x} > \frac{x}{\sin x};\ x \in \left(0, \frac{\pi}{2}\right)$$

$$\sin x + \tan x > 2x;\ x \in \left(0, \frac{\pi}{2}\right),\ \sqrt[n]{n!} > \frac{n}{e};\ n \ge 1$$

$$\frac{\pi}{2} - x \ge \cos x;\ x \in \left(0, \frac{\pi}{2}\right),\ \cos x \ge 1 - \frac{x^2}{2};\ x \in \mathbb{R}$$

$$x - \sin x \le x^3;\ x \in \left(0, \frac{\pi}{2}\right);\ \tan x - x \le x^2;\ x \in [0, 1).$$

$$\frac{2}{\sqrt[k]{n+1} + \sqrt[k]{n-1}} > \frac{1}{\sqrt[k]{n}};\ k \ge 2; n \ge 1,\ \frac{x^2+2}{\sqrt{x^2+1}} \ge 2;\ x \in \mathbb{R};\ \frac{x^2}{1+x^4} \le \frac{1}{2};\ x \in \mathbb{R};$$

$$2\sqrt{x+y} \ge \sqrt{x} + \sqrt{y};\ x + y \ge \sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy};\ \frac{x+y}{4} + \frac{xy}{x+y} - \sqrt{xy} \le \frac{x-2}{2}$$

$$\frac{x}{x^2+yz} \le \frac{1}{4}\left(\frac{1}{y} + \frac{1}{z}\right);\ \frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} \ge \sqrt{x} + \sqrt{y}.$$

Catalan Identity

$x(x-3y-3z)^2+y(3x-y-z)^2+z(3x-y-z)^2=(x+y+z)^3$

Hlawka Inequality

$|x+y+z|+|x|+|y|+|z|\geq|x+y|+|y+z|+|z+x|$

Samuelson Inequality

$\mu-\sigma\sqrt{n-1}\leq x_i\leq\mu+\sigma\sqrt{n-1};\ i\in\overline{1,n}$

$\mu=\left(\sum x_i\right)/n;\ \ \sigma^2=\sum(x_i-\mu)^2/n$

Leibniz Inequality

$a^2+b^2+c^2\leq 9R^2$

Euler Inequality

$R\geq 2r$

Weitzenbock Inequality

$a^2+b^2+c^2\geq 4\sqrt{3}S$

Gerretsen Inequality

$4R^2+4Rr+3r^2\geq p^2$

Mitrinovic Inequality

$p\geq 3\sqrt{3}r$

Hadwiger–Finsler Inequality

$a^2+b^2+c^2\geq(a-b)^2+(b-c)^2+(c-a)^2+4\sqrt{3}S$

Pedoe Inequality

$A^2(b^2+c^2-a^2)+B^2(a^2+c^2-b^2)+C^2(a^2+b^2-c^2)\geq 16Ff$

Schur's Inequality

$p^3-4pq+9r\geq 0;\ p=\sum\cot A;\ \ q=\sum\cot B\cot C;\ \ r=\prod\cot A$

Klamkin Inequality

$$x,y,z\in(0,\infty);\ x\sin A+y\sin B+z\sin C\leq\frac{1}{2}(xy+yz+zx)\sqrt{\frac{x+y+z}{xyz}}$$

Means Inequality – Integral Form

$$m\leq\frac{b-a}{\int_a^b\frac{dx}{f(x)}}\leq e^{\frac{1}{b-a}\int_a^b\ln f(x)dx}\leq\frac{1}{b-a}\int_a^b f(x)dx\leq M;\ f:[a,b]\to[m,M]$$

Cauchy–Schwarz Inequality – Integral Form

$$\int_a^b|f(x)g(x)|\,dx\leq\sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)};f,g:[a,b]\to\mathbb{R}$$

Hölder's Inequality – Integral Form

$$\int_a^b|f(x)g(x)|\,dx\leq\left(\int_a^b|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b|g(x)|^q\,dx\right)^{\frac{1}{q}},\ p>1;\ \frac{1}{p}+\frac{1}{q}=1$$

Minkowski Inequality – Integral Form

$$p \geq 1; f, g : [a, b] \to \mathbb{R}, \left(\int_a^b |f(x)+g(x)|^p\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p\, dx\right)^{\frac{1}{p}}$$

$$\sqrt{\int_a^b [f(x)+g(x)]^2\, dx} \leq \sqrt{\int_a^b f^2(x)dx} + \sqrt{\int_a^b g^2(x)dx}$$

Chebyshev Inequalities – Integral Form

$f, g : [a, b] \to \mathbb{R}$ opposite monotonic functions:

$$\int_a^b f(x)g(x)dx \leq \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right);$$

$f, g : [a, b] \to \mathbb{R}$ monotonic functions of the same directions:

$$\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$$

Jensen's Inequality – Integral Form

$f : [a, b] \to [m, M]$; $\varphi : [m, M] \to \mathbb{R}$, φ convex function:

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b \varphi(f(x))dx$$

Young Inequality – Integral Form

$a \in \mathbb{R}_+$; $b \in f(\mathbb{R}_+)$; $f(0) = 0$;

$f : \mathbb{R}_+ \to \mathbb{R}_+$ continuous increasing function

$$ab \leq \int_a^b f(x)dx + \int_0^b f^{-1}(y)dy$$